

A Novel Switched Linear Observer for Estimating the State of a Dynamical Process with a Mobile Agent

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Abstract—We consider a setting where a mobile agent is tasked with a two-fold objective: (i) persistent surveillance of a geographical region, and (ii) inference of the state of a dynamical process, measurements of which are available at certain points in the region. The main technical challenge introduced by this problem involves dealing with time-varying measurement matrices induced by the motion of the agent. To address this issue, we develop a simple switched linear observer based on a subspace decomposition technique, and analyze its performance for two different classes of patrols. We establish exponential convergence of the estimation error (at any desired convergence rate) under appropriate conditions on the patrols and the observer gains. Our results have implications for the joint design of patrolling and estimation strategies.

I. INTRODUCTION

We consider a setting where a mobile agent seeks to patrol a geographical region and simultaneously estimate the state of a dynamical process evolving over that region. Specifically, we consider an LTI process of the form

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k], \quad (1)$$

where $k \in \mathbb{N}$ is the discrete-time index, $\mathbf{x}[k] \in \mathbb{R}^n$ is the state vector and $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the system matrix. There are N sensing locations spread out over the region of interest. The measurement model at the i -th location is as follows

$$\mathbf{y}_i[k] = \mathbf{C}_i \mathbf{x}[k], \quad (2)$$

where $\mathbf{y}_i[k] \in \mathbb{R}^{r_i}$ and $\mathbf{C}_i \in \mathbb{R}^{r_i \times n}$. We define

$$\mathbf{y}[k] \triangleq [\mathbf{y}_1^T[k] \ \cdots \ \mathbf{y}_N^T[k]]^T, \mathbf{C} \triangleq [\mathbf{C}_1^T \ \cdots \ \mathbf{C}_N^T]^T. \quad (3)$$

We assume that the measurements at the N sensing locations are cumulatively rich enough to allow the state $\mathbf{x}[k]$ to be perfectly reconstructed. Formally, this translates to observability of the pair (\mathbf{A}, \mathbf{C}) .¹ Note, however, that for any given location i , the pair $(\mathbf{A}, \mathbf{C}_i)$ is allowed to be unobservable.

The **objective** of the mobile agent is to persistently visit each of the N locations, and simultaneously estimate the state $\mathbf{x}[k]$ asymptotically. The main technical challenge posed by this problem is as follows. The mobile agent has access to certain measurements only at the sensing locations of the region. Since it persistently commutes between such locations, its measurement model is time-varying. In this context, the main contribution of this paper is to develop

a novel switched linear observer by employing a multi-sensor observable decomposition technique. The key idea is to identify the portion of the state space that can be recovered using the measurements from a given location, and then build a partial observer that reconstructs each such portion. While this idea is inspired by our recent work in distributed state estimation [1], we show that similar concepts are applicable to the design of switched linear observers. We then analyze the performance of the proposed observer for two classes of patrols, namely, periodic and quasi-periodic (patrols that are not necessarily periodic, but have recurring patterns). In each case, we identify conditions on the patrol and the observer structure that guarantee exponential convergence of the estimation error at any desired convergence rate. We point out that the problem of designing patrols in the robotics community [2], [3], and the problem of state estimation using multiple static sensors in the controls community [1], [4]–[6], are individually well-explored. However, to the best of our knowledge, there is limited literature that seeks to address both of these objectives simultaneously using mobile agents. Summarily, this paper makes a preliminary attempt towards gaining a better understanding of the joint design of patrolling and state estimation strategies.

Related Work: In our setting, the mobile agent is aware of its location at all time-steps, and switches between different measurements of the system state (induced by its patrol). This scenario falls under the category of designing observers for switched linear systems where the mode sequence is known. Preliminary work in this area was initiated in [7], with more general results provided in [8]. In comparison with [7], we do not assume that each mode (location) is observable. Since the analysis in [8] pertains to a much more general category of switched linear systems than the one considered in this paper, the design of the observers in [8] are naturally more involved. Specifically, the observer gains in [8] are time-varying and need to be computed in real-time. In contrast, for the specific class of systems considered here, we develop a simple switched linear observer where the observer gains can be computed offline, resulting in an approach that is relatively less computationally-intensive. Also, unlike [8], [9], we focus on switching sequences that model common patrolling schemes, namely periodic and quasi-periodic patrols. Our setting also bears similarities to the sensor scheduling literature where the primary aim is to optimize performance against noise by appropriately choosing the sequence of sensors [10]–[14]. Much of the literature on sensor scheduling focuses on finite-horizon settings [11]–[13], where stability of the error dynamics is

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¹The results obtained in this paper can easily be extended to the more general case when (\mathbf{A}, \mathbf{C}) is detectable.

not a primary concern. A notable exception is the recent work in [14], where the authors study infinite-horizon sensor scheduling, and identify certain specific scheduling policies that lead to a uniformly bounded sequence of covariance matrices. In contrast, the goal of this paper is to identify more general classes of patrols for which stability can be achieved via the construction of suitable switched linear observers.

II. OBSERVER DESIGN

In this section, we develop a novel approach towards designing switched linear observers based on the subspace decomposition technique recently proposed in [1]. Our observer design approach can be summarized as follows. We first order the N locations (the ordering is arbitrary). For the i -th location, we build a partial observer that aims to reconstruct the portion of the state that is unobservable w.r.t. to the cumulative measurements available at the first $i - 1$ locations, but observable w.r.t. the measurements available at the i -th location. In other words, the simple rationale guiding our observer design procedure is to construct partial observers that recover the “innovations” available at each location. Piecing together the innovations available at each location then allows the mobile agent to recover the entire state (under the assumption that (\mathbf{A}, \mathbf{C}) is observable). To proceed, we recall the following result from [1].

Lemma 1. *Given a system matrix \mathbf{A} , and a set of N sensor observation matrices $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_N$, define $\bar{\mathbf{C}} \triangleq [\mathbf{C}_1^T \ \dots \ \mathbf{C}_N^T]^T$. Suppose $(\mathbf{A}, \bar{\mathbf{C}})$ is observable. Then, there exists a similarity transformation matrix \mathbf{T} that transforms the pair $(\mathbf{A}, \bar{\mathbf{C}})$ to $(\bar{\mathbf{A}}, \bar{\mathbf{C}})$, such that*

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{N1} & \mathbf{A}_{N2} & \dots & \mathbf{A}_{N(N-1)} & \mathbf{A}_{NN} \end{bmatrix}, \quad (4)$$

$$\bar{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{C}}_1 \\ \bar{\mathbf{C}}_2 \\ \vdots \\ \bar{\mathbf{C}}_N \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{0} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{C}_{N1} & \mathbf{C}_{N2} & \dots & \mathbf{C}_{N(N-1)} & \mathbf{C}_{NN} \end{bmatrix},$$

and the pair $(\mathbf{A}_{ii}, \mathbf{C}_{ii})$ is observable $\forall i \in \{1, 2, \dots, N\}$. \square

Using the matrix \mathbf{T} given by Lemma 1, we perform the coordinate transformation $\mathbf{x}[k] = \mathbf{T}\mathbf{z}[k]$ to obtain

$$\begin{aligned} \mathbf{z}[k+1] &= \bar{\mathbf{A}}\mathbf{z}[k], \\ \mathbf{y}_i[k] &= \bar{\mathbf{C}}_i\mathbf{z}[k], \quad \forall i \in \{1, \dots, N\}, \end{aligned} \quad (5)$$

where $\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $\bar{\mathbf{C}}_i = \mathbf{C}_i\mathbf{T}$ are given by (4). Commensurate with the structure of $\bar{\mathbf{A}}$ in (4), the vector $\mathbf{z}[k]$ assumes the following form:

$$\mathbf{z}[k] = \begin{bmatrix} \mathbf{z}^{(1)}[k]^T & \dots & \mathbf{z}^{(N)}[k]^T \end{bmatrix}^T, \quad (6)$$

where $\mathbf{z}^{(i)}[k]$ will be referred to as the i -th sub-state. As pointed out earlier, the main technical challenge in estimating $\mathbf{x}[k]$ arises from the fact that the mobile agent can access the measurements of a given location at only those instants in time when it is at the corresponding location. We model this phenomenon using the function $\sigma : \mathbb{N} \rightarrow \mathcal{J} =$

$\{1, \dots, N, \omega\}$. Here, \mathbb{N} represents the set of non-negative integers, and $\sigma[k]$ defines the location of the mobile agent at time-step k . Specifically, for $i \in \{1, \dots, N\}$, $\sigma[k] = i$ implies that the mobile agent is at location i at time-step k , whereas $\sigma[k] = \omega$ implies that the mobile agent is commuting between locations at time-step k . Given this setup, we propose the following partial observer to be employed by the mobile agent for estimating the i -th sub-state $\mathbf{z}^{(i)}[k]$:

$$\hat{\mathbf{z}}^{(i)}[k+1] = \mathbf{F}_{\sigma[k]}^{(i)}\hat{\mathbf{z}}^{(i)}[k] + \sum_{j=1}^{i-1} \mathbf{G}_{\sigma[k]}^{(ij)}\hat{\mathbf{z}}^{(j)}[k] + \beta_{\sigma[k]}\mathbf{L}_i\mathbf{y}_i[k], \quad (7)$$

where $\hat{\mathbf{z}}^{(i)}[k]$ represents the estimate of sub-state $\mathbf{z}^{(i)}[k]$ maintained by the mobile agent, and

$$\begin{aligned} \mathbf{F}_{\sigma[k]}^{(i)} &= \begin{cases} (\mathbf{A}_{ii} - \mathbf{L}_i\mathbf{C}_{ii}) & \text{if } \sigma[k] = i, \\ \mathbf{A}_{ii} & \text{if } \sigma[k] \neq i, \end{cases} \\ \mathbf{G}_{\sigma[k]}^{(ij)} &= \begin{cases} (\mathbf{A}_{ij} - \mathbf{L}_i\mathbf{C}_{ij}) & \text{if } \sigma[k] = i, \\ \mathbf{A}_{ij} & \text{if } \sigma[k] \neq i, \end{cases} \\ \beta_{\sigma[k]} &= \begin{cases} 1 & \text{if } \sigma[k] = i, \\ 0 & \text{if } \sigma[k] \neq i. \end{cases} \end{aligned} \quad (8)$$

In the above equations, \mathbf{L}_i represents an output-injection gain that needs to be appropriately constructed based on the type of patrol being undertaken by the mobile agent, along with other design considerations. Based on the proposed observer, we notice that the mobile agent employs the following strategy for estimating the “innovations” $\mathbf{z}^{(i)}[k]$ available at a given location i . It uses a Luenberger-type update rule when at location i , and updates $\hat{\mathbf{z}}^{(i)}[k]$ in an open-loop manner when not at location i . The coupling of the i -th sub-state with the previous $i - 1$ sub-states is accounted for via the matrices $\mathbf{G}_{\sigma[k]}^{(ij)}$. It should be noted that estimating $\mathbf{z}[k]$ amounts to estimating $\mathbf{x}[k]$, since $\mathbf{x}[k] = \mathbf{T}\mathbf{z}[k]$. Thus, we focus on estimating $\mathbf{z}[k]$. In Sections III and IV, we study the performance of the observer given by (7) and (8) as a function of the movement pattern of the mobile agent.

III. ANALYSIS FOR PERIODIC PATROLS

We first consider the case when the mobile agent visits each location periodically. Specifically, a periodic patrol is characterized by non-negative integers τ_i and positive integers T_i , such that for each $i \in \{1, \dots, N\}$, we have $\sigma[\tau_i + rT_i] = i, \forall r \in \mathbb{N}$. Here, τ_i represents the first time location i is visited, and T_i represents the time-period for location i . We say that a periodic patrol is **feasible** if it satisfies the following constraints: (i) the mobile agent cannot be at more than one location at any given point in time, (ii) each location is visited infinitely often, and (iii) a given location i is visited at time-step k if and only if $k = \tau_i + rT_i$, for some $r \in \mathbb{N}$.² The goal of this section is to address the following question: Suppose the mobile agent follows a feasible periodic patrol and implements the observer given by (7) and (8). Under what conditions on the patrol and the

²Notice that the first two constraints place certain limitations on the values that τ_i and T_i can take on. For instance, to satisfy the second property, we must have $T_i \neq 1, \forall i \in \{1, \dots, N\}$ (assuming $N > 1$).

observer will it be able to estimate $\mathbf{x}[k]$ asymptotically? To this end, we require the following result.

Lemma 2. *Consider an observable pair (\mathbf{A}, \mathbf{C}) , where \mathbf{A} is non-singular. Let the j -th eigenvalue of \mathbf{A} be expressed in the form $\lambda_j = r_j e^{i\theta_j}$, where r_j represents the magnitude of λ_j , θ_j represents its phase, and $i = \sqrt{-1}$. Given a positive integer \bar{T} , the pair $(\mathbf{A}^{\bar{T}}, \mathbf{C})$ is observable if*

$$\bar{T} \neq \frac{2m\pi}{|(\theta_l - \theta_j)|}, \forall m \in \{1, 2, \dots\}, \quad (9)$$

for all $\lambda_l, \lambda_j \in sp(\mathbf{A})$ with $r_l = r_j$ and $\theta_l \neq \theta_j$.³ \square

Proof. Since (\mathbf{A}, \mathbf{C}) is observable, so is the pair $(\mathbf{J}, \bar{\mathbf{C}})$ that results from transforming \mathbf{A} to its Jordan form \mathbf{J} . We now investigate observability of the pair $(\mathbf{J}^{\bar{T}}, \bar{\mathbf{C}})$. To this end, suppose \bar{T} satisfies condition (9). Then, it is clear that if $\lambda_l, \lambda_j \in sp(\mathbf{A})$ and $\lambda_l \neq \lambda_j$, then $\lambda_l^{\bar{T}}, \lambda_j^{\bar{T}} \in sp(\mathbf{A}^{\bar{T}})$ and $\lambda_l^{\bar{T}} \neq \lambda_j^{\bar{T}}$. Based on this fact, the observability of $(\mathbf{J}, \bar{\mathbf{C}})$, and the non-singularity of \mathbf{A} , we can conclude that $(\mathbf{J}^{\bar{T}}, \bar{\mathbf{C}})$ is observable by invoking the PBH test [15]. \square

We now state and prove the main result of this section.

Theorem 1. *Given a dynamical system (1), and a measurement model (2), suppose (\mathbf{A}, \mathbf{C}) is observable, and \mathbf{A} is non-singular. Let the motion of the mobile agent be governed by a feasible periodic patrol characterized by the parameters τ_i and T_i , $i \in \{1, \dots, N\}$, such that each time-period T_i meets the condition imposed by (9). Finally, suppose the mobile agent implements the observer given by (7) and (8). Then, one can design the gains $\mathbf{L}_1, \dots, \mathbf{L}_N$ in a way such that the estimation error of the mobile agent converges to zero exponentially fast at any desired convergence rate μ . \square*

Proof. We first prove asymptotic stability of the error dynamics by inducting on the location number i . Let $\mathbf{e}^{(i)}[k] \triangleq \hat{\mathbf{z}}^{(i)}[k] - \mathbf{z}^{(i)}[k]$ denote the error in estimation of sub-state i . Referring to equations (4) and (5), we see that $\mathbf{z}^{(1)}[k+1] = \mathbf{A}_{11}\mathbf{z}^{(1)}[k]$. Based on the observer equations (7) and (8), we then obtain that $\mathbf{e}^{(1)}[k+1] = (\mathbf{A}_{11} - \mathbf{L}_1\mathbf{C}_{11})\mathbf{e}^{(1)}[k]$ when $\sigma[k] = 1$ and $\mathbf{e}^{(1)}[k+1] = \mathbf{A}_{11}\mathbf{e}^{(1)}[k]$ when $\sigma[k] \neq 1$. Finally, noting that $\sigma[\tau_1 + rT_1] = 1, \forall r \in \mathbb{N}$, we obtain the following recursion for $i = 1$:

$$\mathbf{e}^{(1)}[\tau_1 + (\bar{k} + 1)T_1 + 1] = \mathbf{M}_1\mathbf{e}^{(1)}[\tau_1 + \bar{k}T_1 + 1], \quad (10)$$

where $\bar{k} \in \mathbb{N}$ and $\mathbf{M}_1 = (\mathbf{A}_{11} - \mathbf{L}_1\mathbf{C}_{11})\mathbf{A}_{11}^{(T_1-1)} = (\mathbf{A}_{11}^{T_1} - \mathbf{L}_1\mathbf{C}_{11}\mathbf{A}_{11}^{(T_1-1)})$. Using (10) recursively leads to the following error dynamics:

$$\mathbf{e}^{(1)}[\tau_1 + \bar{k}T_1 + 1] = \mathbf{M}_1^{\bar{k}}\mathbf{e}^{(1)}[\tau_1 + 1], \forall \bar{k} \in \mathbb{N}. \quad (11)$$

To establish asymptotic stability of the above error dynamics, we argue that \mathbf{L}_1 can be chosen to make \mathbf{M}_1 Schur stable.

³Here, $sp(\mathbf{A})$ denotes the spectrum of the matrix \mathbf{A} .

To see this, consider the observability matrix \mathbf{O}_1 of the pair $(\mathbf{A}_{11}^{T_1}, \mathbf{C}_{11}\mathbf{A}_{11}^{(T_1-1)})$, given by

$$\mathbf{O}_1 = \begin{bmatrix} \mathbf{C}_{11}\mathbf{A}_{11}^{(T_1-1)} \\ \mathbf{C}_{11}\mathbf{A}_{11}^{(2T_1-1)} \\ \vdots \\ \mathbf{C}_{11}\mathbf{A}_{11}^{(d_1 T_1-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{11}\mathbf{A}_{11}^{T_1} \\ \vdots \\ \mathbf{C}_{11}\mathbf{A}_{11}^{(d_1-1)T_1} \end{bmatrix} \mathbf{A}_{11}^{(T_1-1)}, \quad (12)$$

where d_1 represents the dimension of \mathbf{A}_{11} . Based on Lemma 1, the pair $(\mathbf{A}_{11}, \mathbf{C}_{11})$ is observable. Since $sp(\mathbf{A}_{11}) \subseteq sp(\mathbf{A})$, non-singularity of \mathbf{A} implies non-singularity of \mathbf{A}_{11} . Based on these observations and the fact that \bar{T}_1 satisfies condition (9), we conclude that the pair $(\mathbf{A}_{11}^{\bar{T}_1}, \mathbf{C}_{11})$ is observable by invoking Lemma 2. Finally, the fact that \mathbf{A}_{11} is non-singular leads to the conclusion that \mathbf{O}_1 in (12) has full column rank. In other words, the pair $(\mathbf{A}_{11}^{T_1}, \mathbf{C}_{11}\mathbf{A}_{11}^{(T_1-1)})$ is observable. It then follows that the gain \mathbf{L}_1 can be chosen so as to make \mathbf{M}_1 Schur stable. Thus, the periodically sampled error dynamics given by (10) converges to zero asymptotically. Notice that the amplification of the error norm within any interval of the form $[(\tau_1 + \bar{k}T_1 + 1), (\tau_1 + (\bar{k} + 1)T_1 + 1)]$, $\bar{k} \in \mathbb{N}$, is bounded above by $\alpha_1^{T_1}$, where $\alpha_1 = \max\{\|\mathbf{A}_{11}\|, \|(\mathbf{A}_{11} - \mathbf{L}_1\mathbf{C}_{11})\|\}$.⁴ Since all matrices under consideration have finite norm, and T_1 is finite, the quantity $\alpha_1^{T_1}$ is also finite. Asymptotic stability of the periodic error dynamics (10) then implies that $\lim_{k \rightarrow \infty} \mathbf{e}^{(1)}[k] = \mathbf{0}$.

Suppose $\mathbf{e}^{(i)}[k]$ converges to zero asymptotically for all $i \in \{1, \dots, q-1\}$, where $1 \leq q-1 \leq N-1$. We now argue that $\lim_{k \rightarrow \infty} \mathbf{e}^{(q)}[k] = \mathbf{0}$. To see this, we first observe that the following recursion can be established based on (4), (5), and the observer update equations (7) and (8):

$$\mathbf{e}^{(q)}[\tau_q + (\bar{k} + 1)T_q + 1] = \mathbf{M}_q\mathbf{e}^{(q)}[\tau_q + \bar{k}T_q + 1] + \sum_{i=1}^{q-1} \mathbf{H}_{qi}\mathbf{v}^{(i)}[\bar{k}], \quad (13)$$

where $\mathbf{M}_q = (\mathbf{A}_{qq}^{T_q} - \mathbf{L}_q\mathbf{C}_{qq}\mathbf{A}_{qq}^{(T_q-1)})$, \mathbf{H}_{qi} represents a constant matrix whose specific entries are unimportant, and

$$\mathbf{v}^{(i)}[\bar{k}] = \begin{bmatrix} \mathbf{e}^{(i)}[\tau_q + \bar{k}T_q + 1] \\ \vdots \\ \mathbf{e}^{(i)}[\tau_q + (\bar{k} + 1)T_q] \end{bmatrix}. \quad (14)$$

Using similar arguments as when $i = 1$, one can design \mathbf{L}_q in a way such that \mathbf{M}_q is Schur stable. Moreover, based on the induction hypothesis, we have that $\lim_{\bar{k} \rightarrow \infty} \mathbf{v}^{(i)}[\bar{k}] = \mathbf{0}, \forall i \in \{1, \dots, q-1\}$. It then follows from Input to State stability (ISS) that $\lim_{k \rightarrow \infty} \mathbf{e}^{(q)}[k] = \mathbf{0}$. Thus, the mobile agent can recover $\mathbf{z}[k]$, and hence $\mathbf{x}[k] = \mathbf{T}\mathbf{z}[k]$.

We now argue that the error $\mathbf{e}[k]$ can be made to converge exponentially fast at any desired convergence rate μ . To establish this fact, pick a set of positive scalars $\{\mu_1, \dots, \mu_N\}$ satisfying $\mu_1 < \mu_2 < \dots < \mu_N < \mu$, where $\mu \in (0, 1)$ is the desired convergence rate. Let the observer gain \mathbf{L}_i be chosen such that $\mathbf{M}_i = (\mathbf{A}_{ii}^{T_i} - \mathbf{L}_i\mathbf{C}_{ii}\mathbf{A}_{ii}^{(T_i-1)})$ has all distinct eigenvalues and spectral radius equal to $\mu_i^{T_i}$. Since we have already established observability of the pair $(\mathbf{A}_{ii}^{T_i}, \mathbf{C}_{ii}\mathbf{A}_{ii}^{(T_i-1)})$, existence of such an \mathbf{L}_i is guaranteed.

⁴Throughout the paper, we use $\|\mathbf{G}\|$ to refer to the induced 2-norm of a matrix \mathbf{G} .

Based on this choice of \mathbf{L}_i , there exists a positive scalar m_i such that $\|\mathbf{M}_i^k\| \leq m_i \mu_i^{kT_i}$, where $k \in \mathbb{N}$ [16]. Let $\phi^{(i)}[k_2, k_1]$ denote the state transition matrix of the error state $\mathbf{e}^{(i)}[k]$ over the interval $[k_1, k_2]$, where $0 \leq k_1 < k_2$. We will establish the fact that

$$\|\phi^{(i)}[k_2, k_1]\| \leq c_i \mu_i^{(k_2 - k_1)}, \quad (15)$$

where

$$c_i = m_i \left(\frac{\alpha_i}{\mu_i} \right)^{\tau_i + 1 + 2T_i}, \quad (16)$$

and $\alpha_i = \max\{\|\mathbf{A}_{ii}\|, \|(\mathbf{A}_{ii} - \mathbf{L}_i \mathbf{C}_{ii})\|\}$.⁵ To see why (15) holds for any $i \in \{1, \dots, N\}$, consider an interval of the form $[k_1, k_2]$, where there exist positive integers r_1, r_2 such that $\tau_i + 1 \leq k_1 < \tau_i + r_1 T_i + 1 < \tau_i + r_2 T_i + 1 < k_2$, and r_1 and r_2 are the smallest and largest integers (respectively) satisfying the inequality. We then have that:

$$\phi[k_2, k_1] = \phi[k_2, \tau_i + r_2 T_i + 1] \phi[\tau_i + r_2 T_i + 1, \tau_i + r_1 T_i + 1] \phi[\tau_i + r_1 T_i + 1, k_1]. \quad (17)$$

Taking norms on both sides of the above equation, and using the sub-multiplicative property of the 2-norm, yields:

$$\begin{aligned} \|\phi[k_2, k_1]\| &\leq \|\phi[k_2, \tau_i + r_2 T_i + 1]\| \|\mathbf{M}_i^{(r_2 - r_1)T_i}\| \|\phi[\tau_i + r_1 T_i + 1, k_1]\| \\ &\stackrel{(a)}{\leq} \left(\alpha_i^{(k_2 - (\tau_i + r_2 T_i + 1))} \right) \left(m_i \mu_i^{(r_2 - r_1)T_i} \right) \left(\alpha_i^{((\tau_i + r_1 T_i + 1) - k_1)} \right) \\ &\stackrel{(b)}{\leq} m_i \alpha_i^{2T_i} \mu_i^{(r_2 - r_1)T_i} \\ &\stackrel{(c)}{\leq} \frac{m_i \alpha_i^{2T_i} \mu_i^{(r_2 - r_1)T_i} \left(\mu_i^{((\tau_i + r_1 T_i + 1) - k_1)} \right) \left(\mu_i^{(k_2 - (\tau_i + r_2 T_i + 1))} \right)}{\mu_i^{2T_i}} \\ &= m_i \left(\frac{\alpha_i}{\mu_i} \right)^{2T_i} \mu_i^{(k_2 - k_1)}. \end{aligned} \quad (18)$$

To achieve the above inequalities, we have used the following facts: (i) $((\tau_i + r_1 T_i + 1) - k_1)$ and $(k_2 - (\tau_i + r_2 T_i + 1))$ are each upper-bounded by T_i (this follows from the way r_1 and r_2 are defined), (ii) $\alpha_i > 1$, and (iii) $\mu_i < 1$. Specifically, (a) follows from the way α_i is defined and by recalling that $\|\mathbf{M}_i^k\| \leq m_i \mu_i^{kT_i}$. Inequality (b) follows from facts (i) and (ii), whereas inequality (c) follows from facts (i) and (iii). Using similar arguments, it can be shown that (18) holds for any interval of the form $[k_1, k_2]$ where $\tau_i + 1 \leq k_1 < k_2$. The effect of the initial offset time $\tau_i + 1$ can then be accounted for via the constant $\left(\frac{\alpha_i}{\mu_i} \right)^{(\tau_i + 1)}$, thereby establishing (15).

Analyzing the decay rate of $\mathbf{e}^{(i)}[k]$ amounts to analyzing the decay rate of an exponentially stable system excited by exponentially decaying inputs (the inputs being the errors $\mathbf{e}^{(j)}[k], j \in \{1, \dots, i-1\}$), where the inputs decay faster (since $\mu_j < \mu_i, \forall j \in \{1, \dots, i-1\}$) than $\mathbf{e}^{(i)}[k]$. We again proceed via induction. Suppose there exist constants \bar{c}_j such that $\|\mathbf{e}^{(j)}[k]\| \leq \bar{c}_j \mu_j^k, \forall k \in \mathbb{N}, \forall j \in \{1, \dots, i-1\}$, where $1 \leq i-1 \leq N-1$. The base case when $j = 1$ follows directly from (15) with $\bar{c}_1 = c_1 \|\mathbf{e}^{(1)}[0]\|$, and c_1 given by

⁵Here, we assume that $\alpha_i, m_i > 1, \forall i \in \{1, \dots, N\}$. If not, the constants c_i can be appropriately adjusted. Basically, (15) points to the fact that if the periodic error dynamics for the i -th location decays at a rate $\mu_i^{T_i}$, then the error $\mathbf{e}^{(i)}[k]$ decays at the slower rate μ_i .

(16). Using the variation of constants formula for discrete-time systems [15], we obtain

$$\mathbf{e}^{(i)}[k] = \phi^{(i)}[k, 0] \mathbf{e}^{(i)}[0] + \sum_{t=0}^{k-1} \sum_{j=1}^{i-1} \phi^{(i)}[k, t+1] \mathbf{G}_{\sigma[t]}^{(ij)} \mathbf{e}^{(j)}[t]. \quad (19)$$

Let $\alpha_{ij} = \max\{\|\mathbf{A}_{ij}\|, \|(\mathbf{A}_{ij} - \mathbf{L}_i \mathbf{C}_{ij})\|\}$. Referring to (8), we then have that $\|\mathbf{G}_{\sigma[t]}^{(ij)}\| \leq \alpha_{ij}, \forall t \in \mathbb{N}$. Taking norms on both sides of (19), and using (15), we obtain

$$\begin{aligned} \|\mathbf{e}^{(i)}[k]\| &\leq \left(c_i \|\mathbf{e}^{(i)}[0]\| + \frac{c_i}{\mu_i} \sum_{j=1}^{i-1} \left(\sum_{t=0}^{k-1} \left(\frac{\mu_j}{\mu_i} \right)^t \right) \alpha_{ij} \bar{c}_j \right) \mu_i^k \\ &\leq \underbrace{\left(c_i \|\mathbf{e}^{(i)}[0]\| + \frac{c_i}{\mu_i} \sum_{j=1}^{i-1} \frac{\alpha_{ij} \bar{c}_j}{1 - \left(\frac{\mu_j}{\mu_i} \right)} \right)}_{\bar{c}_i} \mu_i^k. \end{aligned} \quad (20)$$

In the above equations, we used the sub-multiplicative property of the 2-norm, and the fact that $\mu_j < \mu_i, \forall j \in \{1, \dots, i-1\}$. Noting that \bar{c}_i in (20) is finite, we arrive at the conclusion that $\|\mathbf{e}^{(i)}[k]\| \leq \bar{c}_i \mu_i^k < \bar{c}_i \mu^k$, since $\mu_i < \mu, \forall i \in \{1, \dots, N\}$. We thus obtain:

$$\|\mathbf{e}[k]\| = \sqrt{\sum_{i=1}^N (\|\mathbf{e}^{(i)}[k]\|)^2} \leq \bar{c} N^{\frac{1}{2}} \mu^k, \quad (21)$$

where $\bar{c} = \max_{1 \leq i \leq N} \bar{c}_i$. This completes the proof. \square

Remark 1. For maintaining an asymptotically correct estimate of the state, the assumption of non-singularity of \mathbf{A} in Theorem 1 is not restrictive. Indeed, if \mathbf{A} contains eigenvalues at zero, then a Jordan decomposition reveals that the states corresponding to such eigenvalues will converge to zero in finite time (see also Appendix B in [4]). \square

Remark 2. If \mathbf{A} has eigenvalues of distinct magnitude, then notice that condition (9) will be satisfied by any positive integer \bar{T} . In other words, our analysis reveals that imposing more structure on the system matrix \mathbf{A} allows one to design the patrolling and estimation strategies independently. \square

IV. ANALYSIS FOR QUASI-PERIODIC PATROLS

Based on the third property of a feasible periodic patrol, notice that the mobile agent cannot stay at a given location more than a single time-step. However, for the purpose of surveillance, one might require the agent to spend a finite duration of time at a given location. Additionally, one would ideally like to account for scenarios where periodicity is temporarily lost due to unforeseen circumstances in a dynamic environment. Having said that, one would also like to retain some structure in the movement pattern of the agent for tractability of analysis. Motivated by these concerns, we introduce the notion of a quasi-periodic patrol in this section. We will use the terminology ‘‘the mobile agent visits location i at time-step k ’’ to imply that $\sigma[k] = i$ and $\sigma[k-1] \neq i$. We now define a quasi-periodic patrol as follows.

Definition 1. (Quasi-Periodic Patrol) A quasi-periodic patrol with period T has the following features.

- For each $i \in \{1, \dots, N\}$, there exists $\gamma_i \in (0, 1)$ such that $\sum_{i=1}^N \gamma_i = 1$, and the fraction of time spent by the mobile agent in location i , within every interval of the form $[kT, (k+1)T)$, is given by γ_i . Here, $k \in \mathbb{N}$.
- For each $i \in \{1, \dots, N\}$, there exists $m_i \in (0, \infty)$ and $n_i \in (1, \infty)$, such that the number of visits to location i within every interval of the form $[kT, (k+1)T)$, denoted $f^{(i)}(T)$, satisfies $f^{(i)}(T) \leq m_i T^{\frac{1}{n_i}}$. Here, $k \in \mathbb{N}$.
- The time period T is such that $\gamma_i T$ is a positive integer for each $i \in \{1, \dots, N\}$.

□

Thus, a quasi-periodic patrol is characterized by the period T , and the tuples (m_i, n_i, γ_i) for $i \in \{1, \dots, N\}$. The first property of a quasi-periodic patrol seeks to model the scenario where the fraction of time spent at each location is pre-specified based on surveillance requirements. The second property of a quasi-periodic patrol seeks to model physical constraints that limit the frequency of visits to the different locations. The third property is self-evident in view of the fact that we are dealing with discrete-time systems. We now state and prove the main result of this section.

Theorem 2. *Given a dynamical system (1), and a measurement model (2), suppose (\mathbf{A}, \mathbf{C}) is observable. Let the tuple (m_i, n_i, γ_i) be pre-specified for each $i \in \{1, \dots, N\}$. Then, one can design the gains $\mathbf{L}_1, \dots, \mathbf{L}_N$ and the period T such that (i) all the properties of a quasi-periodic patrol are met, and (ii) the estimation error of the mobile agent based on the observer given by (7) and (8), converges to zero exponentially fast at any desired convergence rate μ . □*

Proof. We proceed by first designing the observer gains $\mathbf{L}_1, \dots, \mathbf{L}_N$ and the period T in an appropriate manner, and then arguing that such a design achieves the desired result.

Design of the Observer Gains \mathbf{L}_i : Given the pre-specified convergence rate $\mu \in (0, 1)$, pick any $\mu^* \in (0, \mu)$. Notice that for each $i \in \{1, \dots, N\}$, there exist positive scalars $c_u^{(i)}$ and $\lambda_u^{(i)}$ such that [16]:

$$\|\mathbf{A}_{ii}^k\| \leq c_u^{(i)} (\lambda_u^{(i)})^k, \forall k \in \mathbb{N}. \quad (22)$$

Pick $\lambda_s^{(i)}$ such that the following inequality is satisfied:

$$\lambda_s^{(i)} \leq \lambda_u^{(i)} \left(\frac{\mu^*}{\lambda_u^{(i)}} \right)^{\frac{1}{\gamma_i}}. \quad (23)$$

We point out that $\lambda_u^{(i)}$ in the above inequality can potentially be strictly less than 1. For each $i \in \{1, \dots, N\}$, we design the observer gain \mathbf{L}_i in a way such that $(\mathbf{A}_{ii} - \mathbf{L}_i \mathbf{C}_{ii})$ has distinct eigenvalues and spectral radius equal to $\lambda_s^{(i)}$. Such a gain \mathbf{L}_i can always be designed in view of the fact that $(\mathbf{A}_{ii}, \mathbf{C}_{ii})$ is observable based on Lemma 1. As a consequence of this design, it is easy to see that for each $i \in \{1, \dots, N\}$, the following inequality is satisfied:

$$\|(\mathbf{A}_{ii} - \mathbf{L}_i \mathbf{C}_{ii})^k\| \leq c_s^{(i)} (\lambda_s^{(i)})^k, \forall k \in \mathbb{N}, \quad (24)$$

where $c_s^{(i)} = \frac{\sigma_{\max}(\mathbf{P}^{(i)})}{\sigma_{\min}(\mathbf{P}^{(i)})}$, and $\mathbf{P}^{(i)}$ diagonalizes $(\mathbf{A}_{ii} - \mathbf{L}_i \mathbf{C}_{ii})$.

Design of the time-period T : For each $i \in \{1, \dots, N\}$, let $\bar{c}^{(i)} = \max\{c_s^{(i)}, c_u^{(i)}\}$. Notice that $\bar{c}^{(i)} \geq 1$ since $c_s^{(i)} \geq 1$. With $m_i' = 6m_i$, pick the time-period T in a way such that:

$$T \geq \max_{1 \leq i \leq N} \left(\frac{m_i' \ln(\bar{c}^{(i)})}{\ln\left(\frac{\mu}{\mu^*}\right)} \right)^{\frac{n_i}{(n_i-1)}}, \quad (25)$$

and $\gamma_i T$ is a positive integer $\forall i \in \{1, \dots, N\}$.

With the observer gains and the time-period chosen as above, we now proceed to analyze the estimation error of the mobile agent whose motion is governed by a quasi-periodic patrol and whose estimation strategy is governed by (7) and (8). We begin by noting that the estimation error $\mathbf{e}[k] = \hat{\mathbf{z}}[k] - \mathbf{z}[k]$ evolves as follows:

$$\underbrace{\begin{bmatrix} \mathbf{e}^{(1)}[k+1] \\ \vdots \\ \mathbf{e}^{(N)}[k+1] \end{bmatrix}}_{\mathbf{e}[k+1]} = \underbrace{\begin{bmatrix} \mathbf{F}_{\sigma[k]}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{G}_{\sigma[k]}^{(21)} & \mathbf{F}_{\sigma[k]}^{(2)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_{\sigma[k]}^{(N1)} & \mathbf{G}_{\sigma[k]}^{(N2)} & \cdots & \mathbf{F}_{\sigma[k]}^{(N)} \end{bmatrix}}_{\mathcal{M}_{\sigma[k]}} \underbrace{\begin{bmatrix} \mathbf{e}^{(1)}[k] \\ \vdots \\ \mathbf{e}^{(N)}[k] \end{bmatrix}}_{\mathbf{e}[k]}, \quad (26)$$

where $\mathcal{M}_{\sigma[k]}$ is a time-varying block lower-triangular matrix with entries given by (8), and $\mathbf{e}^{(i)}[k] = \hat{\mathbf{z}}^{(i)}[k] - \mathbf{z}^{(i)}[k]$. Let us now focus our attention on the evolution of the error dynamics $\mathbf{e}^{(i)}[k]$ at a fixed location $i \in \{1, \dots, N\}$, over a time-interval of the form $[kT, (k+1)T]$. We immediately observe that $\mathbf{e}^{(i)}[k]$ evolves based on a switched linear system model with two modes, namely, a mode corresponding to $\sigma[k] = i$, and a mode corresponding to $\sigma[k] \neq i$. For the switched linear system corresponding to location i , let the switching points over the interval $[kT, (k+1)T]$ (for some $k \in \mathbb{N}$) be given by $k_1 < k_2 < \dots < k_{q_i}$, and let the i_j -th mode be activated during the interval $[k_j, k_{j+1})$, where $j \in \{0, \dots, q_i\}$, $k_0 = kT$ and $k_{q_i+1} = (k+1)T$ (notice that when focusing on the i -th location, i_j corresponds to one of the two modes discussed earlier).⁶ Let $\phi^{(i)}[(k+1)T, kT]$ denote the state transition matrix of the error state $\mathbf{e}^{(i)}[k]$ over the interval $[kT, (k+1)T]$. Based on (26), we obtain:

$$\begin{aligned} \phi^{(i)}[(k+1)T, kT] &= \mathbf{F}_{\sigma[(k+1)T-1]}^{(i)} \cdots \mathbf{F}_{\sigma[kT]}^{(i)} \\ &= \left(\mathbf{F}_{i_{q_i}}^{(i)} \right)^{((k+1)T - k_{q_i})} \cdots \left(\mathbf{F}_{i_0}^{(i)} \right)^{(k_1 - kT)}. \end{aligned} \quad (27)$$

We are led to the following chain of inequalities:

$$\begin{aligned} \|\phi^{(i)}[(k+1)T, kT]\| &\stackrel{(a)}{\leq} (\bar{c}^{(i)})^{(q_i+1)} (\lambda_s^{(i)})^{\gamma_i T} (\lambda_u^{(i)})^{(1-\gamma_i)T} \\ &\stackrel{(b)}{\leq} (\bar{c}^{(i)})^{(q_i+1)} (\mu^*)^T \\ &\stackrel{(c)}{\leq} (\bar{c}^{(i)})^{m_i' T^{\frac{1}{n_i}}} (\mu^*)^T \stackrel{(d)}{\leq} \mu^T. \end{aligned} \quad (28)$$

Prior to justifying the above inequalities, we quickly recall that $(\lambda_u^{(i)})$ and $(\lambda_s^{(i)})$ are given by (22) and (24) respectively, $\bar{c}^{(i)} = \max\{c_s^{(i)}, c_u^{(i)}\}$, and q_i represents the number of switchings w.r.t. the i -th switched linear system within the interval $[kT, (k+1)T]$. Inequality (a) then follows from the

⁶We drop the dependence of the parameters k_j and q_i on k , for clarity.

sub-multiplicative property of the 2-norm, and the fact that based on the properties of a quasi-periodic patrol, the mobile agent spends $\gamma_i T$ fraction of time in location i within the interval $[kT, (k+1)T)$. Inequality (b) follows directly from (23). To arrive at (c), notice that $q_i \geq 1$, since all locations must be visited within a period T (follows from the fact that the fractions γ_i are non-zero). Thus, $q_i + 1 \leq 2q_i$. Also, within a period, the number of switchings for location i is (loosely) upper-bounded by thrice the number of visits to location i , i.e., $q_i \leq 3f^{(i)}(T)$. Combining these two facts and the second property of a quasi-periodic patrol leads to $q_i + 1 \leq 6m_i T^{\frac{1}{n_i}} = m_i' T^{\frac{1}{n_i}}$. Finally, inequality (d) follows directly from (25). Now consider the evolution of the error dynamics (26) over a period T as follows:

$$\underbrace{\begin{bmatrix} \mathbf{e}^{(1)}[(k+1)T] \\ \vdots \\ \mathbf{e}^{(N)}[(k+1)T] \end{bmatrix}}_{\mathbf{e}[(k+1)T]} = \underbrace{\begin{bmatrix} \phi^{(1)}[(k+1)T, kT] & \mathbf{0} & \cdots & \mathbf{0} \\ * & \phi^{(2)}[(k+1)T, kT] & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \phi^{(N)}[(k+1)T, kT] \end{bmatrix}}_{\phi[(k+1)T, kT]} \underbrace{\begin{bmatrix} \mathbf{e}^{(1)}[kT] \\ \vdots \\ \mathbf{e}^{(N)}[kT] \end{bmatrix}}_{\mathbf{e}[kT]}, \quad (29)$$

where the star notation is used for terms that are unimportant. Since T is finite, notice that for each $i \in \{1, \dots, N\}$, $\phi^{(i)}[(k+1)T, kT]$ belongs to a finite family of matrices Σ_i , with the 2-norm of each matrix in the family bounded above by μ^T based on (28). Thus, the joint spectral radius of Σ_i , say $\sigma_i(\Sigma_i)$, is also upper-bounded by μ^T [17]. Since N and T are both finite, the state transition matrix $\phi[(k+1)T, kT]$ for the periodic error dynamics (29) also belongs to a finite family of matrices, say Σ . Since each element in the family Σ is lower block-triangular, we can directly leverage [18, Proposition 1.5] to conclude that the joint spectral radius of Σ is simply $\max_{1 \leq i \leq N} \sigma_i(\Sigma_i)$, and hence upper bounded by μ^T . Thus, the periodic error dynamics (29) converges to zero exponentially fast at a rate upper-bounded by μ^T [17]. Using similar arguments as in the proof of Theorem 1, one can then argue that the error dynamics (26) has an exponential decay rate upper-bounded by μ . This completes the proof. \square

Remark 3. *Since in our formulation, each mode (location) is in general unobservable, the overall error dynamics evolves based on a switched linear system model where every mode is potentially unstable. Stability results for such classes of systems are relatively limited in the switched system literature. Nonetheless, the subspace decomposition approach adopted in this paper allows us to analyze the overall error dynamics by focusing on the error dynamics of the individual sub-systems (each sub-system corresponds to a location). Specifically, since each sub-system switches between a stable mode and an unstable mode, we borrow ideas from average dwell-time theory [19] in our analysis. Furthermore, the block-triangular structure of the error dynamics matrix allows us to leverage results from joint spectral theory [17], [18].* \square

Remark 4. *Since each pair $(\mathbf{A}_{ii}, \mathbf{C}_{ii})$ is observable by construction, one might consider placing all the eigenvalues of $(\mathbf{A}_{ii} - \mathbf{L}_i \mathbf{C}_{ii})$ at zero. Such a strategy will result in finite time convergence if the mobile agent spends d_i consecutive time-steps at location i , where d_i is the dimension of \mathbf{A}_{ii} . However, such a scenario qualifies as only a special case of*

a quasi-periodic patrol. For a general quasi-periodic patrol, placing the eigenvalues of $(\mathbf{A}_{ii} - \mathbf{L}_i \mathbf{C}_{ii})$ at zero may not yield the desired convergence rate, while the design approach outlined in the proof of Theorem 2 always will. \square

V. CONCLUSION

We studied the problem of estimating the state of a linear dynamical process by a mobile agent that persistently commutes between a finite number of locations. To address the issue of time-varying measurement matrices, we developed a novel switched linear observer and analyzed its performance for periodic and quasi-periodic patrols. In each case, we established exponential convergence (at a desired convergence rate) of the mobile agent's state estimate to the true state of the system. Extensions to stochastic patrols, and scenarios involving multiple mobile agents are of interest.

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